Generation of Cherenkov waves in the flow of a Bose–Einstein condensate past an obstacle

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Abstract

The theory of stationary linear wave patterns generated in a supersonic flow of a Bose–Einstein condensate past a point-like obstacle is developed. It is shown that they are located mainly outside the Mach cone corresponding to infinitely long wavelengths. The shape of wave crests and dependence of amplitude on coordinates far enough from the obstacle are calculated. The results are in good agreement with the results of numerical simulations. The theory gives a qualitative description of experiments with Bose–Einstein condensate flow past an obstacle after the condensate's release from a trap.

1. Introduction

The discovery of Bose condensation of dilute gases confined in magnetic traps opened a new chapter in the investigation of Bose–Einstein condensates (BECs). Earlier, Bose–Einstein condensation was used for the explanation of superfluidity of HeII [1]. In the Landau theory [2, 3] of HeII, a superfluid component can be described by a single ‘condensate wave function’ $\psi = \sqrt{n} \exp(i\phi)$ so that the density n of the condensate (different from the density of the superfluid component) and the superfluid velocity $\mathbf{U}$ are expressed in terms of the ψ function as

$$n = |\psi|^2, \quad \mathbf{U} = \frac{\hbar}{m} \nabla \phi. \quad (1)$$

As was shown in [4], in the vicinity of λ -point (i.e. a temperature phase transition of He from its normal phase HeI to the superfluid phase HeII) the ψ -function satisfies the phenomenological Ginzburg–Landau equation. However, atoms in fluid He interact strongly with each other and although all mass is superfluid at $T = 0$, only a small part of it belongs to the condensate (see, e.g., [5]).

The situation simplifies drastically in case of superfluidity of a dilute weakly non-ideal Bose gas. Indeed, it is known that in an ideal Bose gas at $T = 0$, all its atoms belong to the condensate with the wave function corresponding to the ground state of a single-particle Hamiltonian. Then, in a weakly non-ideal Bose gas, almost all atoms belong to the condensate

so that in the first approximation such a gas at $T = 0$ is described by a single condensate wave function. This idea was introduced by Bogoliubov in [6] for the description of elementary excitations in a weakly non-ideal uniform Bose gas, and it was extended to a non-uniform non-stationary state of Bose gas by Gross [7] and Pitaevskii [8] who showed that the wave function ψ of condensate in a non-uniform gas satisfies the equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + V(\mathbf{r})\psi + g|\psi|^2\psi, \quad (2)$$

where

$$g = \frac{4\pi\hbar^2 a_s}{m} \quad (3)$$

is an effective coupling constant related to the s -wave scattering length a_s of atoms, and V is the potential of external forces (e.g., the trap potential).

If we consider a uniform Bose gas with the density n_0 with repulsive interatomic interaction, then its excitations in the form of plane linear waves $\psi \propto \exp[i(kx - \omega t)]$ satisfy the Bogoliubov dispersion relation

$$\omega(k) = \sqrt{c_s^2 k^2 + \left(\frac{\hbar k^2}{2m}\right)^2}, \quad (4)$$

where

$$c_s = \sqrt{\frac{gn_0}{m}} \quad (5)$$

Figure 1. Numerically calculated wave pattern generated by the flow of a BEC past an obstacle. Oblique dark solitons studied in [15] are generated inside the Mach cone. Cherenkov radiation is located outside the Mach cone.

is the speed of sound in a long wave limit $k \rightarrow 0$. The group velocity corresponding to the dispersion law (4) is always greater than the sound velocity (5), and hence such waves can be excited by a supersonic motion of obstacles through the condensate. Besides these excitations, there also exist vortices [9, 10] and dark solitons [11] which can move with subsonic velocities only, and all these excitations were observed in experiments on condensates of dilute gases (see, e.g. [12] and references therein).

When we consider a condensate with several spatial dimensions, plane waves generated by a source can interfere with each other, and hence form interference patterns. Apparently, just such patterns have been observed in recent experiments [13, 14] on free flow of BECs, released from the trap, past an obstacle. The obstacle was generated by a laser beam detuned from the absorption line of atoms so that the BEC was repulsed from the laser beam region. Theoretical investigations [15, 16] showed that in this case two kinds of wave structures appear—a nonlinear structure associated with dark solitons located inside the Mach cone and a linear wave pattern outside the Mach cone where the Mach number is defined by the relation

$$M = \frac{U}{c_s}, \quad (6)$$

where U is the velocity of the flow and c_s is the sound velocity in the limit of infinite wavelength. Then, the Mach (or Cherenkov) cone lines are defined by the condition

$$\sin \theta = \frac{1}{M}, \quad (7)$$

where θ is the angle between the flow direction and the Mach cone line. Example of such a pattern is shown in figure 1.

The process of Cherenkov radiation by the flow of BEC past an obstacle was discussed theoretically in [17], where the drag force acting on the impurity moving with supersonic velocity was calculated. In a recent paper [14], some simple formulae for wavelength were derived and results of numerical simulations were compared with the experiment. Cherenkov radiation in the flow of a two-component BEC was studied

numerically in [18]. In [19], a more detailed theory was developed where the shape of a wave crest was obtained and compared with results of numerical simulations. In the present paper, we develop a more general approach which permits one to describe both the geometric form of the wave crests and the amplitude of the wave generated by a stationary flow of a BEC past an obstacle.

2. Stationary wave pattern

If the size of the obstacle is small compared to the size of the BEC, then we can consider the flow as homogenous and uniform with constant speed U directed along the x -axis and constant density n_0 at $x \rightarrow -\infty$; hence, the Mach number is equal to $M = U/c_s$. The characteristic dimension in a BEC is characterized by a healing length parameter

$$\xi = \frac{\hbar}{\sqrt{2mgn_0}} = \frac{\hbar}{\sqrt{2mc_s}}. \quad (8)$$

Therefore, we introduce the non-dimensional variables

$$\begin{aligned} \tilde{\mathbf{r}} &= \frac{\mathbf{r}}{\sqrt{2}\xi}, & \tilde{t} &= \frac{c_s}{\sqrt{2}\xi} t, & \tilde{n} &= \frac{n}{n_0}, \\ \tilde{U} &= \frac{U}{c_s}, & \tilde{\psi} &= \frac{\psi}{\sqrt{n_0}}, \end{aligned} \quad (9)$$

where $|\tilde{U}| = M$ and the non-dimensional potential of the obstacle

$$\tilde{V}(\tilde{\mathbf{r}}) = \frac{1}{mc_s^2} V(\sqrt{2}\xi\tilde{\mathbf{r}}). \quad (10)$$

Then, the GP equation (2) takes the standard non-dimensional form

$$i\psi_t + \frac{1}{2}\Delta\psi - |\psi|^2\psi - V\psi = 0, \quad (11)$$

where tildes are omitted for the convenience of notation. The flow of a BEC past a point-like obstacle (i.e. with a size much less than the healing length) was studied in [17], and here we shall use the perturbation theory method of that paper for the calculation of the interference pattern created by a supersonic flow past a point-like obstacle with the δ -function potential

$$V(\mathbf{r}) = V_0\delta(\mathbf{r}). \quad (12)$$

If we neglect the influence of the obstacle, then the ψ -function of a uniform BEC with density $n = 1$ depends on time in the reference frame with condensate at rest as $\psi \propto \exp(-it)$. We exclude this dependence by introduction of the Ψ -function according to the definition $\psi = \Psi \cdot \exp(-it)$ so that Ψ satisfies the equation

$$i\Psi_t + \frac{1}{2}\Delta\Psi + [1 - |\Psi|^2]\Psi = 0. \quad (13)$$

In the same reference frame, the obstacle moves with the velocity $-U$ and generates waves in the BEC, which in the linear approximation are described by a small correction $\delta\Psi$ to the unperturbed wave function: $\Psi \approx 1 + \delta\Psi$. Hence, $\delta\Psi$ satisfies the equation

$$i\delta\Psi_t + \frac{1}{2}\Delta\delta\Psi - (\delta\Psi + \delta\Psi^*) - V_0\delta(\mathbf{r} + U\mathbf{t}) = 0, \quad (14)$$

where we have added the potential of the obstacle.

In a stationary case which we are interested in, the wave pattern moves with the obstacle, i.e., in this reference frame we have $\Psi = \Psi(\mathbf{r} + \mathbf{U}t)$ and

$$\frac{\partial}{\partial t} \delta\Psi(\mathbf{r} + \mathbf{U}t) = (\mathbf{U}\nabla)\delta\Psi(\mathbf{r} + \mathbf{U}t).$$

Introducing $\mathbf{r}' = \mathbf{r} + \mathbf{U}t$, we arrive at the equation

$$i(\mathbf{U}\nabla)\delta\Psi + \frac{1}{2}\Delta\delta\Psi - (\delta\Psi + \delta\Psi^*) - V_0\delta(\mathbf{r}) = 0, \quad (15)$$

where primes are omitted. Actually, this equation describes a stationary wave pattern generated by the flow in the reference frame attached to the obstacle and what follows corresponds to just this reference frame.

Equation (15) can be solved by the Fourier method. We introduce the Fourier transform of the wave function:

$$\delta\Psi = \int \delta\Psi_{\mathbf{k}} e^{i\mathbf{k}\mathbf{r}} \frac{d^D k}{(2\pi)^D}, \quad \delta\Psi^* = \int \delta\Psi_{\mathbf{k}}^* e^{-i\mathbf{k}\mathbf{r}} \frac{d^D k}{(2\pi)^D}, \quad (16)$$

where $D = 2, 3$ is the dimension of space and obtain

$$-(\mathbf{k}\mathbf{U} + k^2/2 + 1)\delta\Psi_{\mathbf{k}} - \delta\Psi_{-\mathbf{k}}^* = V_0. \quad (17)$$

Another equation is obtained by means of substitution $\mathbf{k} \rightarrow -\mathbf{k}$ and complex conjugation:

$$-\delta\Psi_{\mathbf{k}} + (\mathbf{k}\mathbf{U} - k^2/2 - 1)\delta\Psi_{-\mathbf{k}}^* = V_0. \quad (18)$$

Solution of equations (17) and (18) reads

$$\delta\Psi_{\mathbf{k}} = V_0 \frac{k^2/2 - \mathbf{k}\mathbf{U}}{(\mathbf{k}\mathbf{U})^2 - k^2(1 + k^2/4)}. \quad (19)$$

Since

$$\delta n = \delta\Psi + \delta\Psi^* = \int (\delta\Psi_{\mathbf{k}} + \delta\Psi_{-\mathbf{k}}^*) e^{i\mathbf{k}\mathbf{r}} \frac{d^D k}{(2\pi)^D},$$

we arrive at the following expression for the density perturbation in the wave pattern created by the flow of BEC past an obstacle:

$$\delta n = V_0 \int \frac{k^2 e^{i\mathbf{k}\mathbf{r}}}{(\mathbf{k}\mathbf{U})^2 - k^2(1 + k^2/4) + i0} \frac{d^D k}{(2\pi)^D}, \quad (20)$$

where we have introduced an infinitesimal positive imaginary term $+i0$ corresponding to an infinitely slow switching on the obstacle potential.

In the experiment [13, 14], the obstacle potential was created by a laser beam, i.e., the wave pattern was actually two-dimensional. Correspondingly, we take $D = 2$ and introduce the polar coordinates (see figure 2) defining the components of vectors $\mathbf{r}$ and $\mathbf{k}$

$$\begin{aligned} x &= r \cos \chi, & y &= r \sin \chi; \\ k_x &= -k \cos \eta, & k_y &= k \sin \eta. \end{aligned} \quad (21)$$

Simple transformation casts equation (20) to the form

$$\delta n = \frac{V_0}{\pi^2} \int_{-\pi}^{\pi} \int_0^{\infty} \frac{k e^{-ikr \cos(\chi+\eta)} dk d\eta}{k^2 - k_0^2 - i0}, \quad (22)$$

where

$$k_0 = 2\sqrt{M^2 \cos^2 \eta - 1}. \quad (23)$$

Integration over k is discussed in appendix A and yields the formula

$$\delta n = -\frac{2V_0}{\pi} \text{Im} \int_{-\pi/2}^{\pi/2} e^{-ikr \cos(\chi+\eta)} d\eta, \quad (24)$$

where k is determined by equation (23) (index ‘0’ is omitted here). Approximate integration over η can be performed in the two most important regions far enough from the Mach cone and close to it.

Figure 2. Coordinates defining a radius vector $\mathbf{r}$ and a wave vector $\mathbf{k}$, normal to the wave front shown schematically by a curved line.

2.1. Cherenkov wave pattern far from the Mach cone

Far from the Mach cone (and hence from the obstacle), the phase $\mathbf{k}\mathbf{r} = rs$, where

$$s(\eta) = k(\eta) \cos(\chi + \eta), \quad (25)$$

is large, and the integral (24) can be evaluated by a standard method of stationary phase. This condition is fulfilled so far for $r \rightarrow \infty$ provided $|k(\eta) \cos(\chi + \eta)| \gg 1/r$. The equation which determines the point of a stationary phase $\partial s / \partial \eta = 0$ gives relationships for the angles (see figure 2)

$$\tan \mu = \frac{2M^2}{k^2} \sin 2\eta, \quad \tan \chi = \frac{(1 + k^2/2) \tan \eta}{M^2 - (1 + k^2/2)}. \quad (26)$$

Taking into account equation (23), we find

$$\cos \mu = \frac{k^2}{2[(M^2 - 2)k^2 + 4(M^2 - 1)]^{1/2}}. \quad (27)$$

With account of (26), we get the expression for a second derivative of the phase

$$\frac{\partial^2 s}{\partial \eta^2} = 8 \frac{\cos \mu}{k^3} [(M^2 - 2)k^2 + 6(M^2 - 1)]. \quad (28)$$

As a result, the expression for the condensate density (24) takes the form

$$\begin{aligned} \delta n &= V_0 \sqrt{\frac{2k}{\pi r}} \frac{[(M^2 - 2)k^2 + 4(M^2 - 1)]^{1/4}}{[(M^2 - 2)k^2 + 6(M^2 - 1)]^{1/2}} \\ &\times \cos \left(kr \cos \mu - \frac{\pi}{4} \right). \end{aligned} \quad (29)$$

As we see from equation (23), the linear waves exist only in the region

$$-\arccos(1/M) \leq \eta \leq \arccos(1/M) \quad (30)$$

outside the Mach cone.

With the help of equations (26), one can find the shape of the lines of constant phase (wave crests) $s = kr \cos \mu$ in a parametric form

$$\begin{aligned} x &= r \cos \chi = \frac{4s}{k^3} \cos \eta (1 - M^2 \cos 2\eta), \\ y &= r \sin \chi = \frac{4s}{k^3} \sin \eta (2M^2 \cos^2 \eta - 1). \end{aligned} \quad (31)$$

Figure 3. Oscillations of the condensate density in a stationary wave in front of the obstacle: $x < 0$, $y = 0$. The condensate flow corresponds to the Mach number $M = 2\sqrt{2}$. The solid line corresponds to the density perturbations calculated numerically and the dashed line to equation (36).

Small values of η correspond to waves in front of the obstacle. In this case, we have

$$\begin{aligned} x &\cong -\frac{s}{2\sqrt{M^2-1}} + \frac{(2M^2-1)s}{4(M^2-1)^{3/2}}\eta^2, \\ y &\cong \frac{(2M^2-1)s}{2(M^2-1)^{3/2}}\eta, \end{aligned} \quad (32)$$

i.e., the lines of stationary phase take a parabolic form

$$x(y) \cong -\frac{s}{2\sqrt{M^2-1}} + \frac{(M^2-1)^{3/2}}{(2M^2-1)s}y^2. \quad (33)$$

The boundary values $\eta = \pm \arccos(1/M)$ correspond to the lines

$$\frac{x}{y} = \pm \sqrt{M^2-1}, \quad (34)$$

i.e., far from the obstacle, the lines approach the straight lines parallel to those forming the Mach cone (6). These results on the form of wave crests reproduce the formulae obtained by another method in [19].

In the region in front of the obstacle where $y = 0$, $x < 0$, the perturbations of the condensate density take the simplest form. Here, we have

$$k = 2\sqrt{M^2-1}, \quad (35)$$

i.e. the wave length $\lambda = 2\pi/k$ is constant and

$$n_1 = 2V_0 \sqrt{\frac{(M^2-1)^{1/2}}{\pi(2M^2-1)|x|}} \cos\left[-2\sqrt{M^2-1}x - \frac{\pi}{4}\right], \quad (36)$$

$y = 0, \quad x < 0.$

The plot illustrating this dependence is shown in figure 3.

The above formulae permit one to derive expressions for the dependence of the density on y coordinate for a fixed value of x which may be convenient for comparison with the experiment and numerical simulations. Far enough from the Mach cone when equation (29) can be applied, we find the dependence of χ on y from the equation

$$\frac{y}{x} = \tan \chi = \frac{(1+k^2/2) \tan \eta}{M^2 - (1+k^2/2)}, \quad (37)$$

then

$$r(\eta) = \frac{y}{\sin \chi(\eta)}, \quad (38)$$

where $k(\eta)$ and $\mu(\eta)$ are defined by equations (23) and (26). In the limit $y \gg x$, we have $\chi \rightarrow \pi/2$; hence, the denominator on the right-hand side of equation (37) vanishes and

$$k \cong \sqrt{2(M^2-1)} \quad \text{for} \quad y \gg x. \quad (39)$$

Comparison with equation (23) gives the limiting value of η ,

$$\cos \eta \cong \frac{\sqrt{M^2+1}}{\sqrt{2}M}. \quad (40)$$

Substitution of these values of the parameters into equation (29) yields

$$\begin{aligned} \delta n(y) &\cong -V_0 \sqrt{\frac{2M}{\pi(M^2-1)y}} \cos[(M-1/M)y - \pi/4] \\ \text{for} \quad y &\gg x. \end{aligned} \quad (41)$$

2.2. Cherenkov wave pattern near the Mach cone

As was indicated above, the method of a stationary phase used for the derivation of (29) needs the condition $|k(\eta) \cos(\chi + \eta)| \gg 1$. According to (23), at the Mach cone, we have $k \rightarrow 0$ and the necessary condition is not fulfilled. To find a wave pattern near the Mach cone (but far enough from the obstacle), one should return to the investigation of the integral (20) and use another approximation for its estimate the details of which are given in appendix B. The final expression has the form

$$\begin{aligned} \delta n = & -\frac{2V_0}{(M^2-1)^{1/6}[3(x \cos \theta + y \sin \theta)]^{2/3}} \\ & \times \text{Ai}'\left[-\frac{2(M^2-1)^{1/6}}{[3(x \cos \theta + y \sin \theta)]^{1/3}}(-x \sin \theta + y \cos \theta)\right], \end{aligned} \quad (42)$$

Figure 4. Oscillations of the condensate density as a function of the transverse coordinate y at $x = 80$ in the vicinity of the Mach cone located at $y = 46.19$ for the Mach number $M = 2$. The solid line shows the profile described by equation (29) and the dashed line corresponds to equation (42).

where $\sin \theta = 1/M$, $\cos \theta = \sqrt{M^2 - 1}/M$ and Ai' is the derivative of the Airy function.

The profile of the wave in the vicinity of the Mach cone is shown in figure 4. As we see, equation (29) reproduces very well the density profile almost everywhere except in the nearest vicinity of the Mach cone and inside it where the density perturbation decays exponentially according to the behaviour of the Airy function in equation (42).

3. Conclusion

The developed theory gives a detailed description of a stationary wave pattern generated by the flow of BEC past a small obstacle in the region outside the Mach cone. It is created by the interference of linear waves radiated by the supersonic motion of the obstacle in analogy to Cherenkov radiation of charged particles moving through a medium with a super-light velocity. As another analogy, one can mention Kelvin's 'ship waves' produced by a ship moving in still water, although in this case there is no notion of a 'sound velocity'. We suppose that this theory explains at least qualitatively the pattern observed in the experiment [13, 14].

Appendix A

We can represent the integral (22) as a sum

$$\int d\eta = \int_{-\pi/2}^{\pi/2} d\eta + \int_{\pi/2}^{3\pi/2} d\eta,$$

and noticing that the second term after substitution $\eta' = \eta - \pi$ becomes equal to a complex conjugate of the first one, we rewrite it as

$$\delta n = \frac{V_0}{\pi^2} \text{Re} \int_{-\pi/2}^{\pi/2} \int_0^\infty \frac{k e^{-ikr \cos(\chi+\eta)} dk d\eta}{k^2 - k_0^2 - i0}. \quad (\text{A.1})$$

To perform integration over k , we notice that the integrand function has a pole in the first quadrant

$$k = \sqrt{k_0^2 + i0} = k_0 + i0, \quad (\text{A.2})$$

which gives the main contribution into the integral for $\cos(\chi + \eta) < 0$. Indeed, taking a closed contour along the positive real axis of k with added quarter of the circle, which gives no contribution into the integral, and a path along the positive imaginary axis whose contribution

$$\int_0^\infty \frac{k e^{-kr \cos(\chi+\eta)} dk}{k^2 + k_0^2} \propto \frac{1}{r^2}, \quad (\text{A.3})$$

is decreasing with r much faster than the contribution of the residue $r^{-1/2}$ (see (29)), we obtain the formula (24).

Appendix B

We introduce new coordinates along the Mach cone (ξ) and normal to it (τ) (i.e., they are rotated to the angle θ around the origin):

$$x = \xi \cos \theta - \tau \sin \theta, \quad y = \xi \sin \theta + \tau \cos \theta. \quad (\text{B.1})$$

In new coordinates, equation (20) takes the form

$$\delta n = \iint \frac{k^2 V(\mathbf{k}) e^{i(k_\xi \xi + k_\tau \tau)}}{(k_\xi M \cos \theta - k_\tau M \sin \theta)^2 - k^2(1 + k^2/4) + i0} \times \frac{dk_\xi dk_\tau}{(2\pi)^2}. \quad (\text{B.2})$$

Far from the obstacle near a Mach cone, the dependence of the wave pattern on the ξ -coordinate is much slower than the dependence on the τ -coordinate; besides that we have $|k| \ll 1$ here. The main contribution into the integral over k_ξ is due to the pole whose position is determined by the equations

$$\begin{aligned} (k_\xi M \cos \theta - k_\tau M \sin \theta)^2 - k^2(1 + k^2/4) &= 0, \\ k_\xi^2 + k_\tau^2 &= k^2. \end{aligned} \quad (\text{B.3})$$

Their approximate solution for $k_\xi \ll k_\tau \ll 1$ is given by

$$k_\xi = -\frac{k_\tau^3}{8\sqrt{M^2-1}}, \quad (\text{B.4})$$

where we have taken into account equation (7). Integration over k_ξ yields

$$\delta n = \frac{V_0}{2\sqrt{M^2-1}} \frac{\partial}{\partial \tau} \left[\frac{1}{\pi} \int_0^\infty \cos \left(\frac{k_\tau^3 \xi}{8\sqrt{M^2-1}} - k_\tau \tau \right) dk_\tau \right], \quad (\text{B.5})$$

and with account of the integral representation of the Airy function

$$\text{Ai}(z) = \frac{1}{\pi} \int_0^\infty \cos \left(\frac{1}{3} \kappa^3 + z\kappa \right) d\kappa, \quad (\text{B.6})$$

we obtain the following expression for the density oscillations in the vicinity of the Mach cone:

$$\delta n = -\frac{2V_0}{(M^2-1)^{1/6}(3\xi)^{2/3}} \text{Ai}' \left[-\frac{2(M^2-1)^{1/6}}{(3\xi)^{1/3}} \tau \right], \quad (\text{B.7})$$

where Ai' denotes the first derivative of the Airy function with respect of its argument. Returning to x and y coordinates, we get the final expression (42).

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